SEQUENCES AND SERIES

QUICK CONCEPT REVIEW

Definition

An arrangement of numbers $< z_1, x_2, \ldots\ldots, x_n, \ldots\ldots >$ according to definite rule or a set of rules is called a Sequence. The various numbers occurring in a sequence are called its terms. The n^{th} term of the sequence is denoted by x_n. The n^{th} term is also called the General Term of the sequence. For example,

1. The numbers $\langle 1, 4, 9, 16, \ldots\ldots \rangle$ represent a sequence written according to the rule $x_n = n^2$, $n \in \mathbf{N}$.
2. The numbers $\left\langle \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots \right\rangle$ represent a sequence written according to the rule $x_n = \frac{n}{n+1}$, $n \in \mathbf{N}$.
3. The numbers $<1, 3, 5, 7, \ldots\ldots>$ represent a sequence written according to the rule $x_n = 2n - 1$, $n \in \mathbf{N}$.
4. The numbers $<1, 3, 7, 13, 21, \ldots>$ represent a sequence written according to the rule $x_n = n^2 + 1$, $n \in \mathbf{N}$.
5. The numbers $(1, 1, 2, 3, 5, 8, \ldots\ldots)$ represent a sequence written according to the following set of rules $x_1 = x_2 = 1$, $x_n = x_{n-1} + x_{n-2}$, $n > 2$, $n \in \mathbf{N}$. This sequence of numbers is called the Fibonacci sequence.
6. The number $\langle 1.4, 1.41, 1.414, 1.4142, \ldots \rangle$ represent a sequence of successive approximations to the irrational number $\sqrt{2}$.
7. The numbers $\langle 2.3, 2.30, 2.302, 2.3025, \ldots. \rangle$ represent a sequence of successive approximations to the irrational number $\log_e 10$.
8. The numbers $\langle 2, 3, 5, 7, 11, 13, \ldots\ldots \rangle$ represent a sequence of prime numbers. In every sequence it is not always possible to write a specific formula as in examples (6), (7) and (8).

Sequence as a Function on the Set of Natural Numbers

We may define a sequence as a function whose domain is some subset of set of natural numbers N of the type $\{1, 2, 3, \ldots\ldots n\}$ = X (say) to other set of number Y. $f : X \to Y$.

- The ordered set of images in Y given by $\{f(1), f(2), f(3) \ldots. f(n)\}$ is the sequence.

- Sequence containing finite number of terms is called a Finite Sequence and it is Infinite Sequence if contains infinite number of terms.
- In the above function representation, the sequence $\{f(1), f(2), f(3),...\}$ is called a Real Sequence if Y = **R**, *i.e.*, $f(1), f(2), f(3)$....are real numbers and it is called a Complex Sequence if Y = C, *i.e.*, $f(1), f(2), f(3)$,....are complex numbers. The terms of the sequence $f(1), f(2), f(3)$....are respectively called the *first term*, *second term*, *third term* and alternatively may be denoted by x_1, x_2, x_3..... or t_1, t_2, t_3......... or a_1, a_2, a_3,........etc.

Methods of Describing A Sequence

(*i*) A sequence may be described by writing first few terms till the rule for writing down the other terms is evident.

(*ii*) A sequence may be described by giving a formula for its general term (the nth term).

(*iii*) A sequence may be described by specifying first few terms and a formula (or a set of formulae) giving a relation between successive terms. Such a formula is called Recursive Formula (or Recurrence Relation) see the example 5.

(*iv*) Some sequences may not be described by any rule.

Series

If $\langle x_n \rangle = \langle x_1, x_2, x_3 ..... \rangle$ is a sequence, then the expression $x_1 + x_2 + x_3 +$..... is called the Series associated with the given sequence.

Progression

A sequence is said to be a Progression if its terms numerically increase (or numerically decrease) continuously.

1. Arithmetic Progression (A.P.)

The sequence $\langle x_1, x_2, x_3, ..., x_n, ... \rangle$ is called an arithmetic progression (A.P.) if

$$x_2 - x_1 = x_3 - x_2 = ... x_n - x_{n-1} = ...$$

In general $x_{n+1} - x_n$ = constant (say, d) $n \in \mathbf{N}$ (the difference of successive term is constant).

The constant difference d is called the *common difference* of the A.P. If the first term x_1 of the A.P. be taken a. Then the standard form of A.P. is $<a, a + d, a + 2d, ...>$

Formula for General Term of an A.P.

The n^{th} term of the A.P., written in standard form is given by $x_n = a + (n - 1)\,d,\ n \in \mathbf{N}$.

Formula for Sum to n Terms of an A.P.

The sum to first n terms of the Arithmetic series $a + (a + d) + (a + 2d) + ...\{a + (n - 1)d\}$ is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}\left[x_1 + x_n\right]$$

IMPORTANT CHARACTERISTIC OF A.P.

1. $x_n = S_n - S_{n-1}$, $n > 1$ and $x_1 = S_1$
2. If n^{th} term of any sequence is a linear in n, then the sequence is an A.P. If x_n is of the form An + B, then the common difference is A.
3. If sum of n terms of any sequence is quadratic in n of the form $An^2 + Bn$ then the sequence is an A.P. and the common difference is 2A.
4. If $<x_1, x_2, x_3,......,.....x_n>$ is an A.P. Then
   (*a*) n^{th} term x_n is called the last term of A.P. and denoted by l also,
   (*b*) $x_1 + x_n = x_2 + x_{n-1} = ...$
   That is the sum of terms equidistant from beginning and end is constant
   (*c*) $\langle x_1 \pm k, x_2 \pm k, x_3 \pm k, ...\rangle$ is an A.P., where k is constant
   (*d*) $\langle kx_1, kx_2, kx_3, ...\rangle$ is an A.P., where k is non zero constant
   (*e*) $\left\langle \frac{x_1}{k}, \frac{x_2}{k}, \frac{x_3}{k} .... \right\rangle$ is an A.P., where k is non-zero constant.
   (*f*) $<x_p, x_{p+q}, x_{p+2q}, ...>$ is an A.P. for any p and q.
   (*g*) The k^{th} term form end of an A.P. = $(n + 1 - k)^{th}$ term from beginning = $a + (n - k)d$.
   Alternate, k^{th} term from end of an A.P. $= l + (k - 1)(-d)$, where l is the last term.
   (*h*) $x_m = \frac{x_{m-k} + x_{m+k}}{2}$, where $0 \le k \le n - m$
5. If $<x_1, x_2, x_3.......>$ and $<y_1, y_2, y_3.....>$ be two different A.P.'s then
   $<x_1 + y_1, x_2 + y_2, x_3 + y_3.......>$ is an A.P.
   $<x_1 - y_1, x_2 - y_2, x_3 - y_3......>$ is an A.P.
6. If three terms to be selected in A.P., choose them $a - d$, a, $a + d$.
7. If four terms to be selected in A.P., choose them $a - 3d$, $a - d$, $a + d$, $a + 3d$.
8. Three numbers a, b, c are in A.P. if and only if $b - a = c - b$, *i.e.*, if and only if $a + c = 2b$.

Arithmetic Mean (A.M) of two Terms *a* and *b*

If a, A, b are in A.P. then A is called Arithmetic Mean of numbers a and b, we get $A = \frac{a+b}{2}$.

Inserting n Arithmetic Means between two Terms a and b

Let $A_1, A_2, A_3, ..., A_n$ be such that $a, A_1, A_2, ...A_n, b$ is A.P.

Clearly $b = x_{n+2} = a + [(n+2) - 1]d \Rightarrow d = \frac{b-a}{n+1}$

Thus the n arithmetic means between a and b are as follow :

$$A_1 = a + d = a + \frac{b-a}{n+1},\ A_2 = a + 2d = a + \frac{2(b-a)}{n+1};\ .....;\ .....;$$

$$A_n = a + nd = a + \frac{n(b-a)}{n+1}$$

We have $A_1 + A_2 + ... + A_n = n\left(\frac{a+b}{2}\right)$

That is, Sum of n A.M. terms between a and $b = n \times$ A.M. of a and b.

2. Geometric Progression (G.P.)

The sequence $<x_1, x_2, x_3......x_n.......>$ is called a geometric progression (G.P.) if

$$\frac{x_2}{x_1} = \frac{x_3}{x_2} = ... = \frac{x_n}{x_{n-1}}......,$$ where none of $x_1, x_2.....,x_n,......$is zero.

In general $\frac{x_{n+1}}{x_2}$ = constant (say, r), $n \in \mathbf{N}$. [The ratio of successive terms is constant].

The constant ratio r is called the *common ratio* of the G.P. If the first term x_1 of the G.P. be taken as a, then the standard form of G.P. is $<a, ar, ar^2,.....>$

Formula for General Term of a G.P.

The n^{th} term of the G.P. written in standard form is given by $x_n = ar^{n-1}$, $n \in \mathbf{N}$.

Formula for Sum of n Term of a G.P.

The sum of first n terms of the geometric series $a + ar + r^2 + ...+ ar^{n-1}$ is given by

$$S_n = \frac{a(1-r^n)}{1-r},\ \text{if } |r| < 1 \text{ and } S_n = \frac{a(r^n-1)}{r-1},\ \text{if } |r| > 1$$

Note

1. When $r = 1$, $S_n = a + a + a + ...$ upto n terms $= na$.
2. If l is the last term of the G.P., then $S_n = \frac{1r - a}{r - 1}$, $r \neq 1$

Formula for the Sum of Infinite Terms of a G.P.

If $|r| < 1$, the sum of infinite terms (S) of the G.P. $a + ar + ar^2 + ...$to infinity is $S = \frac{a}{1-r}$.

IMPORTANT CHARACTERISTICS OF G.P.

1. No term of a G.P. can be zero.
2. $x_n = S_n - S_{n-1}$, $n > 1$ and $x_1 = S_1$.

3. If < x_1, x_2, x_3..........x_n> is G.P. of n terms, then

(a) $x_1x_n = x_2x_{n-1} = \ldots\ldots\ldots$

That is the product of terms equidistant from beginning and end is constant

(b) <x_1k, x_2k, x_3k,.......> is a G.P.

(c) $\left\langle \frac{x_1}{k}, \frac{x_2}{k}, \frac{x_3}{k} \ldots\ldots\right\rangle$ is a G.P., $k \neq 0$

(d) $x_1^k, x_2^k, x_3^k \ldots\ldots >$ is a G.P.

(e) $\left\langle \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \ldots\ldots \right\rangle$ is a G.P.

(f) <x_p, x_{p+q}, x_{p+2q},......> is a G.P. for any p and q.

(g) The k^{th} term from end of a G.P. = (n +1 – k^{th}) term from beginning = ar^{n-k} Alternate, k^{th} term from end of a G.P. $= l\left(\frac{1}{r}\right)^{k-1}$, where l is the last term.

(h) $|x_m| = \sqrt{x_{m-k} \cdot x_{m+k}}, 0 \leq k \leq n - m.$

4. If <x_1, x_2, x_3,......> and <y_1, y_2, y_3........be two different G.P.'s then <x_1y_1, x_2y_2, x_3y_3,......> and $\left\langle \frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \ldots\ldots\ldots \right\rangle$ are G.P.'s.

5. If <x_1, x_2, x_3,.....> is a G.P. of positive terms then < log x_1, log x_2, log x_3,.....> is an A.P. and vice versa.

6. If <x_1, x_2, x_3,.....> is an A.P. then $< a^{x_1}, a^{x_2}, a^{x_3}, \ldots\ldots >$ is a G.P. for some $a > 0$, $a \neq 1$.

7. If three terms to be selected in G.P. choose them $\frac{a}{r}, a\ ar$.

8. If four terms to be selected in G.P. choose then $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

9. Three numbers a, b, c are in G.P. if and only if $\frac{b}{a} = \frac{c}{b}$ or if and only if $b^2 = a.c.$

Geometric Mean (G.M.) of two Terms a and b

If a, G.P. (a and b are positive), then G is the Geometric Mean of numbers a and b.

We get $G = \sqrt{ab}$.

Inserting n Geometric Mean between two Terms a and b

Let a and b be positive numbers. Let G_1, G_2......, G_n be such that a, G_1, G_2,...., G_n, b is a G.P.

Then $b = x_{n+2} = ar^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Thus the n geometric means between a and b are as follows :

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} ; G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} ; G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

We have, $G_1G_2G_3.........G_n = (ab)^{n/2} = \left(\sqrt{ab}\right)^n = G^n$

That is **Product of n geometric mean terms between a and b = n^{th} power of G.M. between a and b.**

Arithmetico-Geometric Sequence

Consider an A.P. $\langle a, a+d, a+2d, ...\rangle$

Consider a G.P. $\langle b, br, br^2, ...\rangle$

If a sequence is formed by multiplying the corresponding terms of above two sequences we get

$\langle ab, (a+d)br, (a+2d)\, br^2, ...\rangle$

This sequence is called an arithmetico-geometric sequence (A.G.S.)

The general term of this sequence is given by $x_n = [a + (n-1)d]\, br^{n-1}$.

Summation of n Term of an A.G.S.

Let $S_n = ab + (a+d)\, br + (a+2d)\, br^2 + ... + [a + (n-1)d]\, br^{n-1}$

$r.S_n = abr + (a+d)\, br^2 + ... + [a + (n-2)d]\, br^{n-1} + [a + (n-1)d]\, br^n$

Subtract, $(1-r)\, S_n = ab + [dbr + dbr^2 + ... + dbr^{n-1}] - [a + (n-1)\, d]\, br^n$

$$(1-r)S_n = ab + \frac{dbr\left(1 - r^{n-1}\right)}{1-r} - \left[a + (n-1)d\right]br^n$$

$$\therefore \quad S_n = \frac{ab}{1-r} + \frac{dbr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{\left[a+(n-1)d\right]br^n}{1-r}$$

Summation of Infinite Terms of an A.G.S.

If $|r| < 1$, then the sum S of infinite terms of the A.G.S. given in standard from can be obtained as following

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{ab}{1-r} + \frac{abr\left(1-r^{n-1}\right)}{(I-r)^2} - \frac{\left\{a+(n-1)d\right\}br^2}{1-r}\right]$$

$$= \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}\left[\lim_{n\to\infty} r^n = 0, \text{ if } |r| <|\right]$$

Sum to n Terms of Special Sequences

1. Sum of first n natural numbers

$$\sum_{r=1}^{n} r = \sum n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

2. Sum of squares of the first n natural numbers

$$\sum_{r=1}^{n} r^2 = \sum n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Sum of cubes of the first n natural numbers

$$\sum_{r=1}^{n} r^3 = \sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. Sum of sequences using sigma natation

If a sequence is characterized by $<x_n>$. Then $S_n = x_1 + x_2 + ... + x_n = \Sigma x_n$

If the general term x_n is given by

$x_n = an^3 + bn^2 + cn^2 + d + k^n$, where $a, b, c, d, k,$ are constants

Then $S_n = \Sigma x_n = \Sigma(an^3 + bn^2 + cn + d + k^n)$

$= a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + d\Sigma 1 + \Sigma k^n$

$$= a\left[\frac{n(n+1)}{2}\right]^2 + b\left\{\frac{n(n+1)(2n+1)}{6}\right\} + c\left[\frac{n(n+1)}{2}\right] + dn + \left(k + k^2 + k^3 + ... + k^n\right)$$

The last bracket being a G.P. whose sum can be found using standard formula.

5. **Difference Series** : Consider $S = x_1 + x_2 + x_3 + ... + x_n$ and let $x_n - x_{n-1} = t_{n-1}$

If $t_1, t_2, ..., t_n$ in an A.P. or G.P. the series $x_1 + x_2 + x_3 + ... + x_n$ is termed as difference series.

To evaluate sum, we proceed as follows :

$S = x_1 + x_2 + x_3 + ... + x_{n-1} + x_n$

$S = x_1 + x_2 + ... + x_{n-2} + x_{n-1} + x_n$

$O = x_1 + [(x_2 - x_1) + (x_3 - x_2) + ... + (x_n - x_{n-1}] - x_n$

$\Rightarrow x_n = x_1 + [t_1 + t_2 + ... + t_{n-1}] \Rightarrow x_n = x_1 + S_{n-1}$

Where S_{n-1} is the sum of series $\{t_1 + t_2 + ...t_{n-1}\}$ which is A.P. or G.P.

Hence x_n may by evaluated and then the sum S may be calculated by $S = \sum x_n$.

6. Method of Difference

Consider the series $S_n = x_1 + x_2 + x_3 + ... + x_n$

Suppose the general term x_k can be expressed as $x_k = f(k) - f(k \pm 1)$.

Then by putting $k = 1, 2, 3, ...n$ we get

$S_n = x_1 + x_2 + x_3 + ... + x_n$

$= f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + ... + f(n) - f(n-1)$

$= f(n) - f(0)$

or $S_n = f(1) - f(2) + f(2) - f(3) + f(3) - f(4) + ... + f(n) - f(n+1)$

$= f(1) - f(n+1)$

Hence, the desired sum can be obtained by putting the values of $f(n)$ and $f(0)$ or $f(1)$ and $f(n+1)$.

ILLUSTRATIVE EXAMPLES

1. The sum to infinite terms of the series $1 + (1+b)r + (1+b+b^2)r^2 + ...$; b and r are proper fractions is :

   (a) $\frac{1}{(1-r)(1-br)}$ (b) $\frac{1}{(1-r)(1-b)}$

   (c) $\frac{1}{(1+r)(1+br)}$ (d) none of these.

   Sol. [See the Difference Series)

   Let $S = 1 + (1+b)r + (1+b+b^2)r^2 + ...$

   $$\frac{S.r = r + (1+b)r^2 + ...}{S(1-r) = 1 + br + b^2r^2 + ... \text{ to infinity}}$$

   $$\therefore S(1-r) = \frac{1}{1-br} \Rightarrow S = \frac{1}{(1-r)(1-br)}$$

2. The sum of the following series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$ upto 16 terms is

   (a) 456 (b) 446

   (c) 452 (d) 444

   Sol. Here, $x_n = \frac{1^3+2^3+3^3+....n^3}{1+3+5+....\text{to } n \text{ terms}}$

   $$= \frac{\frac{n^2(n+1)^2}{4}}{\frac{n}{2}\{2+(n-1)\times 2\}} = \frac{n^2(n+1)^2}{4n^2} = \frac{n^2+2n+1}{4}$$

   $$\therefore S_n = \Sigma x_n \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}\Sigma 1$$

   $$= \frac{1}{4}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}.\frac{n(n+1)}{2} + \frac{1}{4}.n$$

   $$\therefore S_{16} = \frac{1}{4}.\frac{16.17.33}{6} + \frac{1}{2}.\frac{16.17}{2} + \frac{1}{4}.16 = 446$$

3. If z_1 and z_2 are two fixed points then locus of z, satisfying $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

4. The complex equation $|z - z_1| + |z - z_2| = 2a$ where $2a > |z_1 - z_2|$, 'a' is positive real number represents ellipse in complex plane, z_1 and z_2 are affices of two ellipse. If $2a = |z_1 - z_2|$, then $|z - z_1| + |z - z|$ represents the line segment joining z_1 and z_2.

   If $2a < |z_1 - z_2|$, then the equation does not represent any curve.

5. The complex equation $\left||z - z_1| - |z_1 - z_2|\right| = 2a$ where $2a < |z_1 - z_2|$, and 'a' is positive real number, represents a hyperbola in complex plane; z_1 and z_2 are affices of two foci of hyperbola.

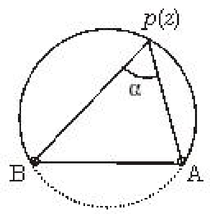
   If $2a = |z_1 - z_2|$, $\left||z - z_1| - |z - z_2|\right| = 2a$ represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB.

6. The complex equation $\left|\dfrac{z - z_1}{z - z_2}\right| = K$ represents a circle if $K \neq 1$ and a straight line if $K = 1$.

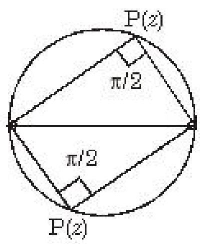
7. The complex equation $|z - z_1|^2 + |z - z_2|^2 = K$ represents a circle if $K \geq \dfrac{1}{2}|z_1 - z_2|^2$, K a real number.

8. Let z_1 and z_2 be two fixed points and α be a real number such that $0 \leq \alpha \leq \pi$ then

   (*a*) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$, $0 < \alpha < \pi$, $\alpha \neq \dfrac{\pi}{2}$ represents a segment of the circle passing through $A(z_1)$ and $B(z_2)$.

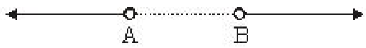


   (*b*) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{2}$ represent a circle with diameter as the segment $A(z_1)$ and $B(z_2)$.



(*c*) $\arg\left(\frac{z - z_1}{z - z_2}\right) = 0$ represents the line segment joining A(z_1) and B(z_2)

(*d*) $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pi$ represents the straight line joining A(z_1) and B(z_2) but excluding the segment AB



QUESTIONS FOR TEACHERS

1. The P + R, 9th and r^{th} terms of an A.P. are respectively x, y and z, then the value of x (9 – r) + y (r – p) + z (p – 9) is

   (*a*) $x + y + z$ (*b*) $p + q + r$

   (*c*) $px + 9y + rz$ (*d*) 0

2. If m arithmetic means are inserted between 1 and 31 so that the ratio of the 7^{th} and $(m - 1)^{th}$ means is 5 : 9, then the value of m is

   (*a*) 9 (*b*) 11

   (*c*) 11 (*d*) 14

3. If S_1, S_2, S_3,...., S_p are the sums of infinite geometric series whose first terms are 1, 2, 3,...., p and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{p+1}$ respectively. Then $S_1 + S_2 + S_3 + \ldots + S_p =$

   (*a*) $\frac{p(p+1)}{2}$ (*b*) $\frac{p(p+3)}{2}$

   (*c*) p (*d*) $\frac{p}{2}$

4. If a, b, c are in A.P. and x, y, z are in G.P. then what is the value of x^{b-c}, y^{c-a}, z^{a-b}?

   (*a*) 1 (*b*) –1

   (*c*) 0 (*d*) 2

5. If the non-zero numbers x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are also in A.P. then

   (*a*) $x = y = z$ (*b*) $xy = yz$

   (*c*) $x^2 = yz$ (*d*) $z^2 = xy$

6. The sum to n terms of the series $\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \ldots$ is

   (*a*) $\frac{3n}{n+1}$ (*b*) $\frac{6n}{n+1}$

(c) $\frac{9n}{n+1}$ (d) $\frac{12n}{n+1}$

7. If $S_n = np + \frac{n}{2}(n-1)\theta$, where S_n denotes the sum of the first n terms of an A.P., then the common difference is

(a) P + Q (b) 2P + 3Q

(c) Q (d) Q – P

8. The value of 0.423 is

(a) $\frac{419}{999}$ (b) $\frac{419}{990}$

(c) $\frac{423}{1000}$ (d) None

9. The sum of n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+ ......$ is

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$

(c) $n + 2^{-n} - 1$ (d) $2^n - 1$

10. The product of n geometric means between two given numbers a and b is

(a) $(ab)^n$ (b) $(ab)^{n/2}$

(c) $(ab)^{2n}$ (d) $(ab)^{3n}$

HINTS AND ANSWERS *(for Teachers)*

1.(d) $x = p$th term $= a + (p-1)\,d$

y = 9th term $= a + (9-1)\,d$

$z = r$th term $= a + (r-1)\,d$

then proceeds :

2.(d) – 0

3.(b) $\frac{p(p+3)}{2}$

Here $S_1 = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

$S_2 = \frac{2}{1-\frac{1}{3}} = 3 ...... S_p = p+1$

$\therefore \quad S_1 + S_2 + ... + S_p = \frac{p}{2}(2+p+1) = \frac{p(p+3)}{2}$

4.(*a*) 1

Hint : If a, b, c are in A.P. $\Rightarrow 2b = a + c$

x, y, z are in G.P. $\Rightarrow y^2 = xz$

then $x^{b-c}\, y^{c-a}\, z^{c-b} = x^0 z^0 = 1$

5.(*a*) $x = y = z$

Hint : x, y, z are in A.P.

$\Rightarrow 2y = x + z$ ....(*i*)

$\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P.

$\Rightarrow 2\tan^{-1}y = \tan^{-1}x + \tan^{-1}z$

$$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-xz} \Rightarrow 1 - y^2 = 1 - xz$$

$\Rightarrow y^2 = xz \Rightarrow x, y, z$ are in G.P ....(*ii*)

From (*i*) and (*ii*), we get $x = y = z$

6.(*b*) $\dfrac{6n}{n+1}$

Hint : $$\text{Tr} = \frac{2r+1}{1^2 + 2^2 + \ldots\ldots + r^2} = 6\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

So required sum $$= \sum_{r=1}^{n} Tr = 6\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{6n}{n+1}$$

7.(*c*) Q Let an be the n^{th} term of the A.P. then

an $= S_n - S_{n-1} = P = (n-1)\,Q$

$\therefore$ C.D. $= a_n - a_{n-1} = Q$

8.(*b*) $\dfrac{419}{990}$

Hint : S = 0.423 then

S = 0.42323232323....

$= 0.4 + 0.023 + 0.00023$

$= 0.4 + 23 \times 10^{-3} + 23 \times 10^{-5} + \ldots$

$$= 0.4 + \frac{23 \times 10^{-3}}{1 - 10^{-2}} = \frac{419}{990}$$

9.(*c*) $n + 2^{-n} - 1$

Hint : Sum $= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) + ... + n$ terms

$$= n - \frac{1}{2}\frac{\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = n - 1 + 2^{-n}$$

10.(*b*) $(ab)^{\frac{n}{2}}$

QUESTIONS FOR STUDENTS

1. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 =$

   (*a*) $\frac{n(n+1)(2n+1)}{6}$ (*b*) $\frac{n(n+1)}{2}$

   (*c*) $\left(\frac{n(n+1)^2}{2}\right)$ (*d*) $\frac{n(n+1)(n+2)}{6}$

2. In a G.P. the $(p+q)^{th}$ term is a and $(p-q)^{th}$ term is b, then the p^{th} term is

   (*a*) $a + b$ (*b*) $a - b$

   (*c*) $\sqrt{ab}$ (*d*) ab

3. If abc are in G.P. then $a^2b^2c^2\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) =$

   (*a*) $a^3 + b^3 + c^3$ (*b*) $a^2 + b^2 + c^2$

   (*c*) $a + b + c$ (*d*) $ab + bc + ca$

4. If one geometric mean G and two arithmetic means p and q be inserted between two numbers, then G^2 is equal to

   (*a*) $(3p - q)(3q - p)$ (*b*) $(2p - q)(2p - p)$

   (*c*) $(2p - q)(4q - p)$ (*d*) None

5. If the p^{th} term of an A.P. is q and the 9^{th} term is p_1 then its $(p+q)^{th}$ term is

   (*a*) $p + q$ (*b*) $p + q + 1$

   (*c*) $p + q - 1$ (*d*) 0

6. The ratio of the sum of n terms of two A.P.'s is $(3n - 13) : (5n + 21)$. then ratio of 24^{th} terms of the two progression is

   (*a*) 1 : 2 (*b*) 2 : 3

   (*c*) 3 : 5 (*d*) 7 : 11

7. If a_1, a_2, a_3..., an in A.P. where $a_i > 0$ for all i, then

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+.....+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=$$

(a) $\frac{n+1}{\sqrt{a_1}+\sqrt{a_n}}$ (b) $\frac{n}{\sqrt{a_1}+\sqrt{a_n}}$

(c) $\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$ (d) 0

8. 100^{th} terms of the series 1 + 3 + 7 + 15 +...... is

(a) $2^{50}-1$ (b) $2^{80}-1$

(c) $2^{100}-1$ (d) $2^{25}-1$

9. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p + q)$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ (d) 2

10. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of its progression equals

(a) $\sqrt{5}$ (b) $\frac{1}{2}\left(\sqrt{5}-1\right)$

(c) $\frac{1}{2}\left(1-\sqrt{5}\right)$ (d) $\frac{1}{2}\left(\sqrt{5}+1\right)$

11. Let a_1, a_2, a_3......be terms of A.P. If $\frac{a_1+a_2+....+a_p}{a_1+a_2+....+a_q}=\frac{p^2}{q^2}$, $p \neq q$ then $\frac{a_6}{a_21}$ equals

(a) $\frac{41}{11}$ (b) $\frac{7}{2}$

(c) $\frac{2}{7}$ (d) $\frac{11}{41}$

12. If $x=\sum_{n=0}^{\infty}a^n$, $y=\sum_{n=0}^{\infty}b^n$, $z=\sum_{n=0}^{\infty}c^n$ where a, b, c are in A.P. and $|a| < 1$, $|b| < 1$, $|c| < 1$ then x, y, z are in

(a) A.P. (b) G.P.

(c) Arithmetic-Geometric progression (d) None

13. If in A.P. the $p + h$ term is $\frac{1}{q}$ and $q + h$ term is $\frac{1}{p}$, then the sum of pq terms is

(a) $\frac{1}{pq}$ (b) $\frac{1}{p}+\frac{1}{q}$

(c) $(p+q)\left(\frac{1}{p}+\frac{1}{q}\right)$ (d) $\frac{pq+1}{2}$

14. If l, m, n are the p^{th}, q^{th} and r^{th} term of a G.P. all positive then $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} =$

    (a) -1 (b) 0

    (c) 1 (d) 2

15. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. if $a < b < c$ and $a+b+c=\frac{3}{2}$, then the value of a is

    (a) $\frac{1}{2}-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}-\frac{1}{\sqrt{3}}$

    (c) $\frac{1}{2\sqrt{3}}$ (d) $\frac{1}{2\sqrt{2}}$

16. An infinite G.P. has first term 'x' and sum '5' then x belongs to

    (a) $x < -10$ (b) $-10 < 0$

    (c) $< x < 10$ (d) None of these

17. If $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ are in A.P. then the expression $a_1, a_2 + a_2a_3 + \ldots\ldots + a_{n-1}\, a_n$ is equal to

    (a) $n(a_1 - a_n)$ (b) $(n-1)\,(a_1 - a_n)$

    (c) na_1an (d) $(n-1)\,a_1a_n$

18. If in a series $S_n = an^2 + bn + c$, where S_n denotes the sum of n terms, then

    (a) The series is always arithmetic.

    (b) The series is arithmetic from the second term onwards.

    (c) The series may or may not be arithmetic.

    (d) The series cannot be arithmetic.

19. The value of x in $(0, \pi)$ which satisfy the equation $g\ |+|\cos x + \cos^2 x + |\cos^3 x| + \ldots + 0\ \infty = 4^3$ is

    (a) $\left\{\frac{\pi}{2}, \frac{3\pi}{4}\right\}$ (b) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

    (c) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (d) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

20. If S_1, S_2, S_3,......, S_n are the sum of infinite geometric series whose first terms are 1, 2, 3,...., n and whose common ratio are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n+1}$ respectively, then value of $S_1^2 + S_2^2 + S_3^2 + \ldots + S_{2n-1}^2$ is equal to

(a) $\frac{1}{3}\left[n(2n+1)(4n+1)-3\right]$ (b) $\frac{1}{3}\left[n(2n+1)(4n+1)+3\right]$

(c) $\frac{1}{3}\left[n(2n-1)(4n+1)-3\right]$ (d) $\frac{1}{3}\left[n(2n+1)(4n-1)-3\right]$

21. The 20^{th} terms of the series 2 + 3 + 5 + 9 + 16 +.....is

(a) 950 (b) 975

(c) 990 (d) 1010

22. $(\underbrace{666....6}_{n-\text{digits}})^2 + (\underbrace{888....8}_{n-\text{digits}})$ is equal to

(a) $\frac{4}{9}\left(10^n - 1\right)$ (b) $\frac{4}{9}\left(10^{2n} - 1\right)$

(c) $\frac{4}{9}\left(10^n - 1\right)^2$ (d) $\frac{4}{9}\left(10^{2n} + 1\right)$

23. If z, y, z are in G.P. and $z^x = b^y = c^z$ then

(a) $\log_b{}^a = \log_a{}^c$ (b) $\log_c{}^b = \log_a{}^c$

(c) $\log_b{}^a = \log_a{}^b$ (d) None

24. The sum of n terms of the series 1 + (1 + 3) + (1 + 3 + 5) +.....is

(a) n^2 (b) $\left[\frac{n(n+1)}{2}\right]^2$

(c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{n(n+1)^2}{4}$

25. If x, y, z are $p + h$, $q + h$ and $r + h$ terms respectively of an A.P. and also of G.P. then $x^{y-z} y^{x-y}$ is equal to

(a) xyz (b) 0

(c) 1 (d) -1

26. If A and G be the A.M. and G.M. respectively between two members then the numbers are

(a) $A \pm \sqrt{G^2 - A^2}$ (b) $A \pm \sqrt{A^2 - G^2}$

(c) $A \pm \sqrt{A^2 + G^2}$ (d) $G \pm \sqrt{A^2 - G^2}$

27. If the interior angles of a polygon are in A.P. with common difference 5° and smallest angle is 120°, then the number of sides of the polygon is

(a) 9 or 16 (b) 9

(c) 16 (d) 13

28. If the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are in G.P. then

(a) $c^3a = b^3d$ (b) $ca^3 = bd^3$

(c) $c^3b = c^3d$ (d) $ab^3 = cd^3$

29. The sum of all possible products of the first n natural numbers taken two by two is

(a) $\frac{1}{24} n(n+1)(n-1)(3n+2)$ (b) $\frac{n(n+1)(2n+2)}{6}$

(c) $\frac{n(n+1)(n-1)(2n+3)}{24}$ (d) None of these

30. If $S = 1 + a + a^2 + \ldots$ to ∞ $(a < 1)$, then the value of a is

(a) $\frac{S}{S-1}$ (b) $\frac{S}{1-S}$

(c) $\frac{S-1}{S}$ (d) $\frac{1-S}{S}$

ANSWERS

1.(d) $\frac{n(n+1)(n+2)}{6}$ 2.(c) $\sqrt{ab}$

3.(a) $a^3 + b^3 + c^3$ 4.(b) $(2p-q)(2q-p)$

5.(d) 0 6.(a) 1 : 2

7.(c) $\frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$ 8.(c) $2^{100} - 1$

9.(c) $\sqrt{2}$ 10.(b) $\frac{1}{2}\left(\sqrt{5} - 1\right)$

11.(d) $\frac{11}{41}$ 12.(d) None

13.(d) $\frac{pq+1}{2}$ 14.(b) 0

15.(a) $\frac{1}{2} - \frac{1}{\sqrt{2}}$ 16.(d) None of these

17.(d) $(n-1)\, a, a_n$ 18.(b) The series is arithmetic from the second term onwards.

19.(c) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ 20.(a) $\frac{1}{3}\left[n(2n+1)(4n+1) - 3\right]$

21.(c) 990 22.(b) $\frac{4}{9}\left(10^{2n} - 1\right)$

23.(c) $\log_b{}^a = \log_c{}^b$ 24(c) $\frac{n(n+1)(2n+1)}{6}$

25(c) 1 26.(b) $A \pm \sqrt{A^2 - G^2}$

27.(b) 9 28.(a) $c^3 a = b^3 d$

29.(a) $\frac{1}{24} n(n+1)(n-1)(3n+2)$ 30.(c) $\frac{S-1}{S}$